Discrete Mathematics 310 (2010) 6–11

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Optimal 1-planar graphs which triangulate other surfaces

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ARTICLE INFO

Article history:
Received 30 December 2008
Received in revised form 14 July 2009
Accepted 16 July 2009
Available online 31 July 2009

Keywords:
Optimal 1-planar graph
Closed surface
Triangulation

ABSTRACT

We show that, for any given non-spherical orientable closed surface F^2, there exists an optimal 1-planar graph which can be embedded on F^2 as a triangulation. On the other hand, we prove that there does not exist any such graph for the nonorientable closed surfaces of genus at most 3.

1. Introduction

A simple graph G is said to be *1-planar* if it can be drawn on the sphere S^2(or the plane) so that each of its edges crosses at most one other edge. The drawing is regarded as a continuous map $f : G \to S^2$ which may not be injective. To simplify our notation, we often consider that a given 1-planar graph G is already mapped on the sphere, and denote its image by G itself. An edge is said to be *crossing* if it crosses another edge in a 1-planar graph G, and to be *non-crossing* otherwise.

This 1-planar graphs were first considered by Ringel [11] in the problem of the simultaneous coloring of the vertices and faces of plane graphs. His conjecture was that every 1-planar graph can be colored by at most 6 colors, and Borodin [2] proved it. The class of 1-planar graphs has been studied from several viewpoints in many papers, for example [1,4,13]. Recently, the author discussed re-embeddability of 1-planar graphs in [16].

For a 1-planar graph G, we have an inequality $|E(G)| \leq 4|V(G)| - 8$ where $V(G)$ and $E(G)$ are its vertex set and edge set, respectively. (This inequality was proved in some papers. For example, see [4].) A 1-planar graph G is said to be *optimal* if $|E(G)| = 4|V(G)| - 8$. It had already proved in [16] that every optimal 1-planar graph G is obtained by adding a pair of crossing edges to each face of a 3-connected quadrangulation on the sphere, hence every vertex of G has even degree.

A *triangulation* of a closed surface is a simple graph cellularily embedded on the surface, so that each face is triangular, while a *quadrangulation* of a closed surface is a graph of which each face is bounded by a 4-cycle. (A *k-cycle* means a cycle of length k.) These particular embeddings have been major themes in "topological graph theory" and Negami et al. [7,10,14,15] discussed the existence of graphs having both these properties; i.e., a graph can be embedded into different closed surfaces as a triangulation and as a quadrangulation.

In this paper, we shall discuss the existence of optimal 1-planar graphs which can be embedded on other closed surfaces as triangulations. (Note that such a triangulation is an *even triangulation*; i.e., each vertex has even degree. Recently, Nakamoto et al. have discussed these even triangulations by focusing on a local deformation called an *N-flip*. See [5,8,9].) For example, see Fig. 1. The left-hand side is an optimal 1-planar graph with 8 vertices, while the right-hand side is a triangulation on the torus. (To obtain the torus, identify two horizontal sides and two vertical sides of the rectangle, respectively, in the figure.) By the observation of the adjacency of those two embeddings, we can easily confirm that their abstract graphs are

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doi:10.1016/j.disc.2009.07.016

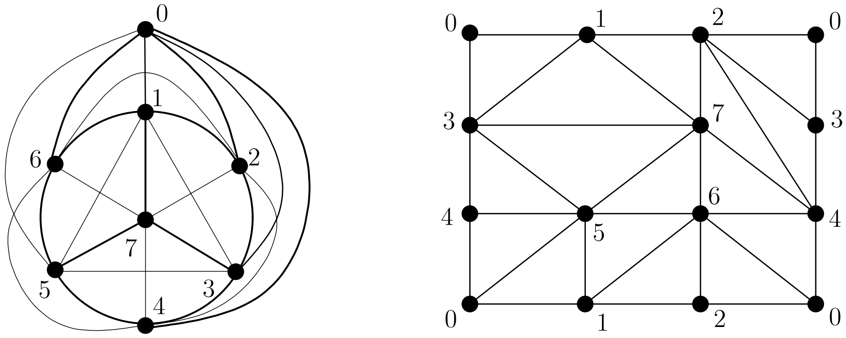

Fig. 1. XW_6 triangulates the torus.

isomorphic to each other. If there exists such an optimal 1-planar graph G that triangulates a closed surface F^2, the numbers of its vertices and edges are given by Euler's formula as follows (where $\chi(F^2)$ is the Euler characteristic of F^2):

$$|V(G)| = 8 - 3\chi(F^2), \quad |E(G)| = 24 - 12\chi(F^2).$$

In this paper, we shall prove the following theorem, showing some methods to construct examples:

Theorem 1. *Given an orientable closed surface F^2, there is an optimal 1-planar graph which can be embedded on F^2 as a triangulation.*

In the next section, we shall introduce the generating theorem of optimal 1-planar graphs. In Section 3, we prove Theorem 1 by constructing concrete examples. At last, we shall discuss those graphs for nonorientable closed surfaces. However, we prove that no optimal 1-planar graph can triangulate a nonorientable surface with genus at most 3.

2. Generating theorem of optimal 1-planar graphs

Let G be an optimal 1-planar graph and let $\{v_1v_3, v_2v_4\}$ be a pair of crossing edges. Since G is optimal and has the largest number of edges, there are four non-crossing edges v_1v_2, v_2v_3, v_3v_4 and v_4v_1 around the crossing edges; this fact is easy to confirm and is mentioned in [4,16]. By the fact, it is easy to see that the removal of all the crossing edges from G yields a simple quadrangulation on the sphere. We call it a *quadrangular subgraph* of G and denote it by $Q(G)$.

In the following lemma, we shall show that there is one-to-one correspondence between the set of optimal 1-planar graphs and the set of 3-connected quadrangulations on the sphere.

Lemma 2. *Let H be a simple quadrangulation on the sphere. Then there exists a simple optimal 1-planar graph G such that $H = Q(G)$ if and only if H is 3-connected.*

Proof. First, we shall prove the necessity. Suppose that H is not 3-connected. Then, there is a separating simple closed curve which intersects with H at exactly two vertices, say u, v, and passes through the inside of two quadrangular faces $aubv$ and $cudv$. When we add pairs of crossing edges for all the faces of H, two edges joining u to v clearly would become multiple edges, a contradiction.

Next, we show the sufficiency. We add a pair of crossing edges to each face of the 3-connected quadrangulation H and assume that the resulting graph has multiple edges. First, suppose a pair of multiple edges consists of an edge $uv \in E(H)$ and an added crossing edge. Then there exists a quadrangular face $uavb$, in which the crossing edge is added. Now we could find a 3-cycle uva in H, however it is contrary to H being bipartite; since every quadrangulation on the sphere is bipartite.

On the other hand, suppose that both multiple edges are added crossing edges. Regard the pair of those multiple edges as a simple closed curve intersecting with H at exactly two points. Then, H clearly could not be 3-connected, contrary to the assumption. ■

Let G be an optimal 1-planar graph and let f be a face of its quadrangular subgraph bounded by 4-cycle $v_1v_2v_3v_4$. For G, we define two pairs of operations (a reduction and an expansion). The first one as a reduction is called the *Q_f-contraction* (at $\{v_1, v_3\}$) and defined as follows: (i) Delete a pair of crossing edges $\{v_1v_3, v_2v_4\}$. (ii) Identify v_1 and v_3 and replace the two pairs of multiple edges $\{v_1v_2, v_3v_2\}$ and $\{v_1v_4, v_3v_4\}$ with two single edges respectively. If the operations break the simpleness of the graph, then we don't apply it. The inverse operation of the Q_f-contraction is said to be the *Q_v-splitting* (see the left-hand side of Fig. 2).

The expansion of the second operation is said to be the *Q_4-addition* which is performed as follows: (i) Delete a pair of crossing edges $\{v_1v_3, v_2v_4\}$. (ii) Add a 4-cycle $\{u_1u_2u_3u_4\}$ inside a 2-cell bounded by $v_1v_2v_3v_4$ and join v_i and u_i for $i = 1, 2, 3, 4$. (iii) Add a pair of crossing edges to each 2-cell bounded by a 4-cycle consisting of non-crossing edges. The inverse operation (reduction) of Q_4-addition is called the *Q_4-removal* (see the right-hand side of Fig. 2). Similarly, we should keep the simpleness of the graph, especially when we apply a 4-cycle removal.

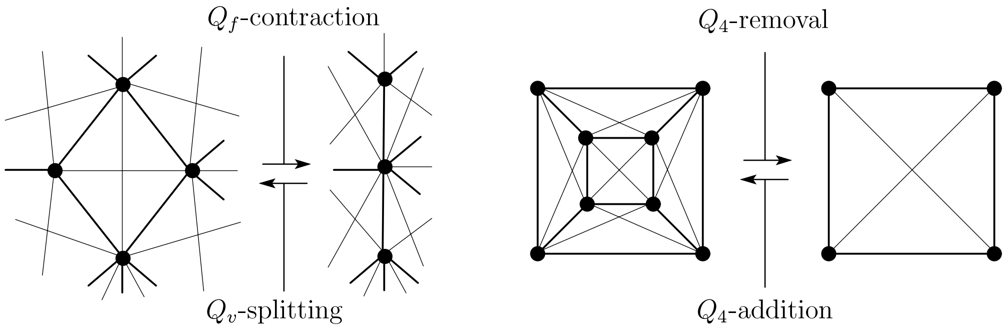

Fig. 2. Two operations for optimal 1-planar graphs.

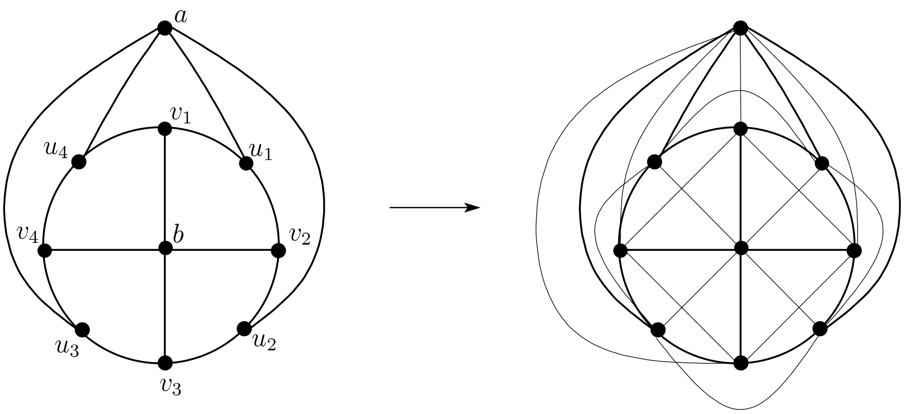

Fig. 3. Pseudo double wheel and *X*-pseudo double wheel.

Embed a $2k$-cycle $v_1u_1v_2u_2\cdots v_ku_k$ into the sphere and put two vertices a and b inside the 2-cell regions separated by the cycle respectively. Next, we add edges av_i and bu_i for $i = 1, \ldots, k$. We call the resulting quadrangulation of the sphere a *pseudo double wheel* and denote by W_{2k} (see the left-hand side of Fig. 3). Since W_2 has multiple edges and W_4 have two vertices of degree 2, the smallest pseudo double wheel is W_6, which is a cube; since we have to guarantee the simpleness and the 3-conectedness of this quadrangulation. We add pairs of crossing edges to all faces of $W_{2k}(k \geq 3)$, and obtain the optimal 1-planar graph. This resulting graph is said to be a *X-pseudo double wheel* and denoted by XW_{2k}. See the right-hand side of Fig. 3.

Now, we show the generating theorem of optimal 1-planar graphs.

Theorem 3. *Every optimal* 1*-planar graph can be obtained from* $XW_{2k}(k \geq 3)$ *by a sequence of* Q_v*-splittings and* Q_4*-additions.*

Proof. In [3], Brinkmann et al. presented the generating theorem of 3-connected quadrangulations on the sphere. In the theorem, they proved that any 3-connected quadrangulation can be constructed from a pseudo double wheel by a sequence of two operations completely corresponding to a Q_v-splitting and a Q_4-addition (when we consider the quadrangular subgraph). Therefore, our theorem immediately follows from Lemma 2. ■

It is easy to see that XW_6 is the minimal optimal 1-planar graph with eight vertices and it can be embedded into the torus as a triangulation according to our expectation (see Fig. 1). Furthermore, this triangulation is uniquely (up to isomorphism) embeddable into the torus by an argument of Sasao[12].

3. Slit *N*-flip sum

Let G_i be an optimal 1-planar graph on the sphere S_i that can be embedded on a surface F_i as a triangulation T_i, and let $\{u_iw_i, v_ix_i\}$ be a pair of crossing edges in G_i. (As mentioned before, there are four non-crossing edges u_iv_i, v_iw_i, w_ix_i and x_iu_i in G_i.) Furthermore, suppose that two 3-cycles $u_iv_iw_i$ and $u_ix_iw_i$ bound faces of T_i sharing an edge u_iw_i for $i = 1, 2$.

Now cut open the two spheres S_1 and S_2, each including G_1 and G_2, along the edges u_1v_1 and u_2v_2, respectively, and paste them along the resulting boundaries so that u_1 is identified with v_2 and v_1 with u_2. Now, note that the resulting graph is an optimal 1-planar graph but has multiple edges u_1v_1. To eliminate the multiple edges, replace the 3-path $w_1u_1v_1w_2$ with another 3-path $w_1x_2x_1w_2$ (see the left-hand side of Fig. 4). Let $G_1 \ddagger G_2$ denote the resulting simple optimal 1-planar graph on the sphere S obtained from the two slitted surfaces. In the similar way we can construct a new triangulation $T_1 \ddagger T_2$

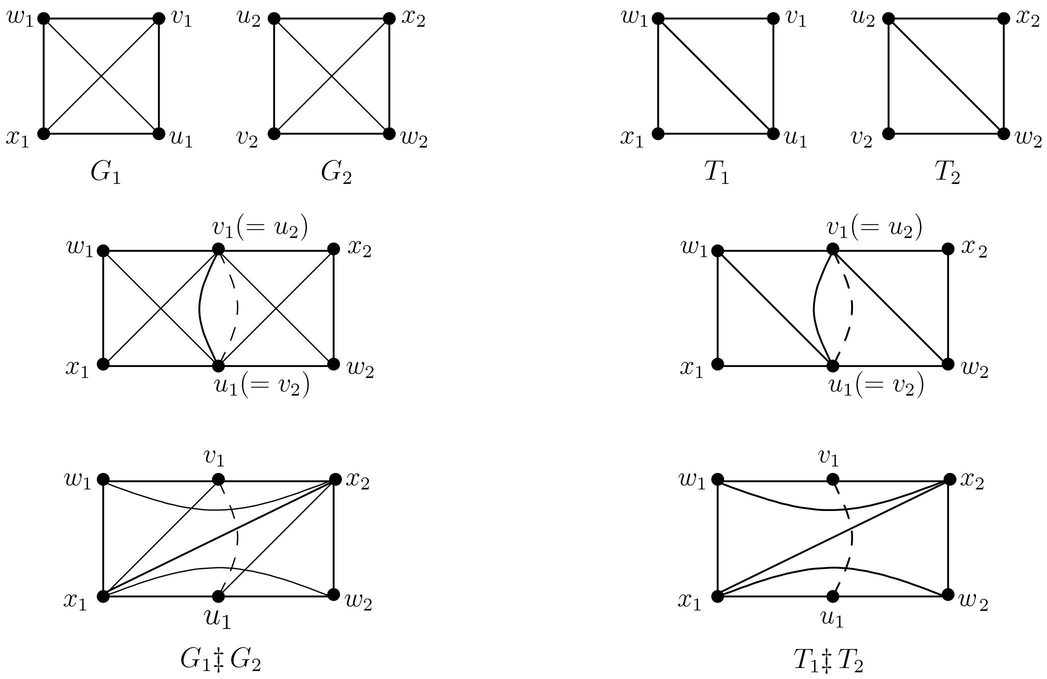

Fig. 4. Slit N-flip sum.

on a closed surface F obtained from F_1 and F_2 slitted along u_1v_1 and u_2v_2 (see the right-hand side of Fig. 4). It is clear that underlying graphs of $G_1 \ddagger G_2$ and $T_1 \ddagger T_2$ are isomorphic. We say that $G_1 \ddagger G_2$ is obtained from G_1 and G_2 (or $T_1 \ddagger T_2$ is obtained from T_1 and T_2) by a *slit N-flip sum*. (As mentioned in the introduction, an *N-flip* is a local deformation which transforms an even triangulation into another even triangulation.) Like $v_1u_1x_1$ (or $v_2u_2x_2$) in the above operation, a 3-path vux of an optimal 1-planar graph G which can be embedded into a closed surface except the sphere as a triangulation T is said to be a *useful corner* if it satisfies the followings: (i) vux forms a corner of a face $vuxw$ of $Q(G)$, (ii) each of two 3-cycles vuw and uwx bounds a face of T.

Lemma 4. *Any graph $G_1 \ddagger G_2$ (and $T_1 \ddagger T_2$) obtained by a slit N-flip sum has a useful corner.*

Proof. See Fig. 4. The 3-path $v_1x_2x_1$ in $G_1 \ddagger G_2$ (and $T_1 \ddagger T_2$) is our required useful corner. ■

Now we shall prove Theorem 1.

Proof of Theorem 1. We shall construct an optimal 1-planar graph that triangulates the orientable closed surface of genus g for every $g \geq 1$. We had already obtained such a graph for $g = 1$, say G_1 (and T_1 as a triangulation on the torus), in Fig. 1. In the figure, observe that the 3-path 734 is a useful corner. We prepare a copy of G_1, say G_2 (and T_2), and carry out a slit N-flip sum to G_1 and G_2 and obtain an optimal 1-planar graph $G_1 \ddagger G_2$ which has an triangular embedding $T_1 \ddagger T_2$ on the double torus with 14 vertices (when $g = 2$). Furthermore by Lemma 4, we can repeat applying slit N-flip sums to obtain the desired genus of the triangulated surface as we want. ■

4. Nonorientable cases

In this section, we discuss the existence of optimal 1-planar graphs which triangulate nonorientable closed surfaces. However, for those surfaces with genus at most 3, we shall give the negative answer. Note that if there exists such a graph G then $|V(G)| = 2 + 3k$ where k is the nonorientable genus of the surface which is triangulated by G.

Proposition 5. *There does not exist an optimal 1-planar graph which can be embedded on the projective plane or the Klein bottle as a triangulation.*

Proof. By Euler's formula, such a graph should have 5 vertices when $k = 1$, and 8 vertices when $k = 2$. By Theorem 3, the X-pseudo double wheel XW_6 with 8 vertices is the unique smallest optimal 1-planar graph and hence we can immediately omit the former. Therefore we consider whether this XW_6 triangulates the Klein bottle.

If it does, XW_6 can triangulate both the torus and the Klein bottle. However, it is impossible by the argument in [6]. In the paper, they discussed the existence of graphs which can be embedded into both the torus and the Klein bottle. There is only one irreducible triangulation on the torus (or the Klein bottle), but the graph is not isomorphic to XW_6. ■

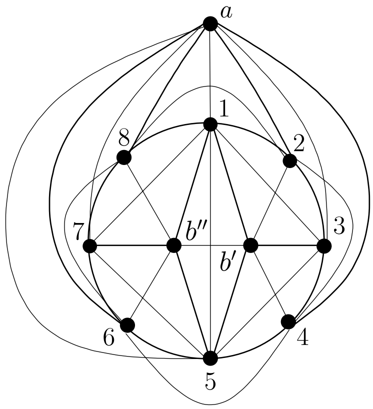

Fig. 5. The unique optimal 1-planar graph with 11 vertices.

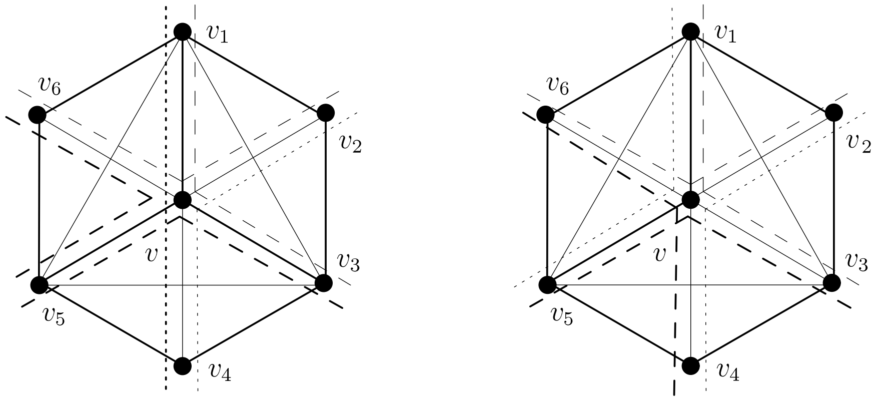

Fig. 6. 3-cycles around v.

The following proposition guarantees that we only have to check the graph in Fig. 5, when the nonorientable genus is 3.

Proposition 6. *There is the unique optimal* 1*-planar graph with 11 vertices as shown in Fig. 5.*

Proof. Let G be an optimal 1-planar graph with 11 vertices. By Theorem 3 again, we can obtain G from either XW_6 or XW_8 by a sequence of Q_v-splittings; since the number of its vertices exceeds 12, if we apply a Q_4-addition even to the minimal graph XW_6.

Observe that any Q_v-splitting to a vertex of degree 6 yields multiple edges in an optimal 1-planar graph and hence we may suppose that G is obtained from XW_8 by exactly one Q_v-splitting. We have only two vertices with degree greater than 6 in XW_8, labeled a and b in Fig. 3. We split b to two vertices b' and b'' so that each of two resulting vertices has degree 6. The resulting graph is shown in Fig. 5. ■

Let G be a simple connected graph. Assume that G has a family of cycles of length 3, denoted by $\mathcal{C}$, such that each edge of G is contained in exactly two distinct cycles of $\mathcal{C}$. Then $\mathcal{C}$ is called a *3-cycle double cover* of G. It is well-known that if the 3-cycles of $\mathcal{C}$ incident to each vertex $v \in V(G)$ induce a cyclic order over the neighbors of v, then G can be embedded into a closed surface as a triangulation so that each 3-cycle in $\mathcal{C}$ bounds a face. Note that the order corresponds to the *rotation* around v in the triangulation.

See Fig. 6. The dotted and dashed angles are those of 3-cycles around a vertex v of G. One can check that the left-hand side shows *good* 3-cycles around v which can be taken as a part of a 3-cycle double cover corresponds to a triangular embedding. On the other hand, the right-hand side does not induce a cyclic order over the whole neighbors of v and we call them *bad* 3-cycles around v.

Proposition 7. *There does not exist an optimal* 1*-planar graph which can be embedded on the nonorientable closed surface of genus* 3 *as a triangulation.*

Proof. By Proposition 6, we only have to check the unique optimal 1-planar graph G with 11 vertices shown in Fig. 5. For a contradiction, suppose that G has a triangular embedding into the nonorientable closed surface of genus 3 and the corresponding 3-cycle double cover $\mathcal{C}$.

First, we consider such 3-cycles around b'. There exist only two 3-cycles which contain the edge $b'b''$ and hence the 3-cycles $1b'b''$ and $5b'b''$ belong to $\mathcal{C}$. Now we have three 3-cycles $15b'$, $12b'$ and $13b'$ which can cover the edge $1b'$. However if we take $15b'$ as such a 3-cycle, then the 3-cycles and $15b'$ become bad 3-cycles around v. Therefore we consider the latter two cases. Assume that we take $13b'$ and consider the second 3-cycle which covers the edge $5b'$ with $5b'b''$. Now, taking $35b'$ immediately makes bad 3-cycles and hence we must adopt $45b'$. Further, since taking $34b'$ under these conditions also yields bad 3-cycles we must take $24b'$ to cover $4b'$. Therefore, we have the following two cases as good 3-cycles around b', up to symmetry: (i) $\{1b'b'', 5b'b'', 13b', 23b', 24b', 45b'\} \subset \mathcal{C}$, (ii) $\{1b'b'', 5b'b'', 12b', 23b', 34b', 45b'\} \subset \mathcal{C}$.

The same argument holds around b'' and we have the following four possibility, up to symmetry, depending on how to take (i) or (ii) around the vertices b' and b'':

(A) $\{1b'b'', 5b'b'', 12b', 23b', 34b', 45b', 56b'', 67b'', 78b'', 81b''\} \subset \mathcal{C}$,
(B) $\{1b'b'', 5b'b'', 12b', 23b', 34b', 45b', 56b'', 68b'', 78b'', 71b''\} \subset \mathcal{C}$,
(C) $\{1b'b'', 5b'b'', 13b', 23b', 24b', 45b', 56b'', 68b'', 78b'', 71b''\} \subset \mathcal{C}$,
(D) $\{1b'b'', 5b'b'', 13b', 23b', 24b', 45b', 57b'', 67b'', 68b'', 81b''\} \subset \mathcal{C}$.

For example, we assume (A). In G, the edge 13 is on the four 3-cycles $13b'$, 123, 135 and $13a$. However, we cannot take $13b'$ since the rotation around b' is already completed by good 3-cycles; for otherwise, the edges $1b'$ and $3b'$ would be covered by $\mathcal{C}$ three times. Moreover, if we take 123 then the three 3-cycles 123, $12b'$ and $23b'$ become bad 3-cycles around the vertex 2. Therefore, the edge 13 should be covered by two 3-cycles 135 and $13a$. By the same argument, the edge 17 should be covered by 157 and $17a$. However, 135, $13a$, 157 and $17a$ form bad 3-cycles around the vertex 1. Therefore, (A) is not the case.

We omit the proofs of remaining three cases (B), (C) and (D) since they are just routines. We can, however, conclude that each of (B), (C) and (D) is not the case, that is, those 3-cycles cannot be extended to a 3-cycle double cover. This is contrary to G having a 3-cycle double cover $\mathcal{C}$. Hence, G cannot triangulate the nonorientable closed surface of genus 3. ■

At the end of the paper, we leave the following problem for the reader.

Problem 1. Does there exist an optimal 1-planar graph which can triangulate a nonorientable closed surface of genus at least 4?

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